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Yueyun Hu

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Summary. Consider a real-valued branching random walk in the boundary case. Using the techniques developed by Aïdékon and Shi [5], we give two integral tests which describe respectively the lower limits for the minimal position and the upper limits for the associated additive martingale.

1 Introduction

Let $\{V(u), u \in \mathbb{T}\}$ be a discrete-time branching random walk on the real line $\mathbb{R}$, where $\mathbb{T}$ is an Ulam-Harris tree which describes the genealogy of the particles and $V(u) \in \mathbb{R}$ is the position of the particle u. When a particle u is at n-th generation, we write $|u| = n$ for $n \geq 0$. The branching random walk V can be described as follows: At the beginning, there is a single particle $\varnothing$ located at 0. The particle $\varnothing$ is also the root of $\mathbb{T}$. At the generation 1, the root dies and gives birth to some point process $\mathcal{L}$ on $\mathbb{R}$. The point process $\mathcal{L}$ constitutes the first generation of the branching random walk $\{V(u), |u| = 1\}$. The next generations are defined by recurrence: For each $|u| = n$ (if such u exists), the particle u dies at the $(n+1)$-th generation and gives birth to an independent copy of $\mathcal{L}$ shifted by $V(u)$. The collection of all children of all u together with their positions gives the $(n+1)$-th generation. The whole system may survive forever or die out after some generations.

Plainly $\mathcal{L} = \sum_{|u|=1} \delta_{\{V(u)\}}$. Assume $\mathbb{E}[\mathcal{L}(\mathbb{R})] > 1$ and that

$$\mathbb{E}\left[\int e^{-x}\mathcal{L}(dx)\right] = 1, \qquad \mathbb{E}\left[\int xe^{-x}\mathcal{L}(dx)\right] = 0. \tag{1.1}$$

When the hypothesis (1.1) is fulfilled, the branching random walk is called in the boundary case in the literature (see e.g. Biggins and Kyprianou [9] and [10], Aïdékon and Shi [5]). Under some integrability conditions, a general branching random walk can be reduced to the boundary case after a linear transformation, see Jaffuel [19] for detailed discussions. We shall assume (1.1) throughout this paper.

Denote by $\mathbb{M}_n := \min_{|u|=n} V(u)$ the minimal position of the branching random walk at generation n (with convention $\inf \emptyset \equiv \infty$). Hammersly [17], Kingman [20] and Biggins [8] established the law of large numbers for $\mathbb{M}_n$ (for any general branching random walk), whereas the second order limits have recently attracted many attentions, see [1, 18, 12, 2] and the references therein. In particular, Aïdékon [2] proved the

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convergence in law of $\mathbb{M}_n - \frac{3}{2}\log n$ under (1.1) and some mild conditions, which gives a discrete analog of Bramson [11]'s theorem on the branching brownian motion.

Concerning the almost sure limits of $\mathbb{M}_n$, there is a phenomena of fluctuation at the logarithmic scale ([18]): Under (1.1) and some extra integrability assumption: $\exists\, \delta > 0$ such that $\mathbb{E}[\mathcal{L}(\mathbb{R})^{1+\delta}] < \infty$ and $\mathbb{E}\big[\int_{\mathbb{R}}(e^{\delta x} + e^{-(1+\delta)x})\mathcal{L}(dx)\big] < \infty$, the following almost sure limits hold:

$$\begin{aligned}\limsup_{n\to\infty} \frac{\mathbb{M}_n}{\log n} &= \frac{3}{2}, \qquad \mathbb{P}^*\text{-a.s.},\\ \liminf_{n\to\infty} \frac{\mathbb{M}_n}{\log n} &= \frac{1}{2}, \qquad \mathbb{P}^*\text{-a.s.},\end{aligned}$$

where here and in the sequel,

$$\mathbb{P}^*(\cdot) := \mathbb{P}\left(\cdot|\mathcal{S}\right),$$

and $\mathcal{S}$ denotes the event that the whole system survives. The upper bound $\frac{3}{2}\log n$ is the usual fluctuation for $\mathbb{M}_n$ because $\mathbb{M}_n - \frac{3}{2}\log n$ converges in law ([2]). It is a natural question to ask how $\mathbb{M}_n$ can approach the unusual lower bound $\frac{1}{2}\log n$.

Aïdékon and Shi [5] proved that under (1.1) and the following integrability conditions

$$\sigma^2 := \mathbb{E}\Big[\int_{\mathbb{R}} x^2 e^{-x}\mathcal{L}(dx)\Big] < \infty, \tag{1.2}$$

$$\mathbb{E}\Big[\eta(\log_+ \eta)^2 + \widetilde{\eta}\log_+ \widetilde{\eta}\Big] < \infty, \tag{1.3}$$

where $\eta := \int_{\mathbb{R}} e^{-x}\mathcal{L}(dx)$, $\widetilde{\eta} := \int_0^\infty x\, e^{-x}\mathcal{L}(dx)$ and $\log_+ x := \max(0, \log x)$, then

$$\liminf_{n\to\infty}\left(\mathbb{M}_n - \frac{1}{2}\log n\right) = -\infty, \qquad \mathbb{P}^*\text{-a.s.}$$

Furthermore, they asked whether there is some deterministic sequence $a_n \to \infty$ such that

$$-\infty < \liminf_{n\to\infty}\frac{1}{a_n}\left(\mathbb{M}_n - \frac{1}{2}\log n\right) < 0, \qquad \mathbb{P}^*\text{-a.s.?}$$

The answer is yes: we can choose $a_n = \log\log n$. Moreover, we can give an integral test to describe the lower limits of $\mathbb{M}_n$:

Theorem 1.1 *Assume* (1.1), (1.2) *and* (1.3). *For any function* $f \uparrow \infty$*,*

$$\mathbb{P}^*\left(\mathbb{M}_n - \frac{1}{2}\log n < -f(n), \quad i.o.\right) = \begin{cases} 0 \\ 1 \end{cases} \iff \int^\infty \frac{dt}{t\exp(f(t))} \begin{cases} < \infty \\ = \infty \end{cases}, \tag{1.4}$$

where i.o. means infinitely often as the relevant index $n \to \infty$.

As a consequence of the integral test (1.4), we have that for any $\varepsilon > 0$, $\mathbb{P}^*$-a.s. for all large $n \geq n_0(\omega)$, $\mathbb{M}_n - \frac{1}{2}\log n \geq -(1+\varepsilon)\log\log n$ whereas there exists infinitely often n such that $\mathbb{M}_n - \frac{1}{2}\log n \leq -\log\log n$. Hence $\mathbb{P}^*$-a.s., $\liminf_{n\to\infty}\frac{1}{\log\log n}(\mathbb{M}_n - \frac{1}{2}\log n) = -1$.

The behaviors of the minimal position $\mathbb{M}_n$ are closely related to the so-called additive martingale $(W_n)_{n\geq 0}$:

$$W_n := \sum_{|u|=n} e^{-V(u)}, \qquad n \geq 0,$$

with the usual convention: $\sum_\emptyset \equiv 0$. By Biggins [8] and Lyons [23], $W_n \to 0$ almost surely as $n \to \infty$. The problem to find the rate of convergence (or a Seneta-Heyde norming) for W_n arose in Biggins and Kyprianou [9] and was studied in [18]. Aïdékon and Shi [5] gave a definite result to this problem. Let

$$D_n := \sum_{|u|=n} V(u) e^{-V(u)}, \qquad n \geq 1, \tag{1.5}$$

be the derivative martingale (which is a martingale under the boundary condition (1.1)). It was shown in Biggins and Kyprianou [9] that $\mathbb{P}$-a.s., D_n converges to some nonnegative random variable D_∞. Moreover under (1.1), (1.2) and (1.3), $\mathbb{P}^*$-a.s., $D_\infty > 0$, as shown in [9] and [2].

Theorem (Aïdékon and Shi [5]). *Assume* (1.1), (1.2) *and* (1.3). *Then under* $\mathbb{P}^*$,

$$\sqrt{n} W_n \overset{(p)}{\to} \sqrt{\frac{2}{\pi\sigma^2}}\, D_\infty,$$

as $n \to \infty$. *Moreover*

$$\limsup_{n\to\infty} \sqrt{n}\, W_n = \infty, \qquad \mathbb{P}^*\text{-a.s.}$$

Furthermore Aïdékon and Shi conjectured that

$$\liminf_{n\to\infty} \sqrt{n}\, W_n = \sqrt{\frac{2}{\pi\sigma^2}}\, D_\infty, \qquad \mathbb{P}^*\text{-a.s.} \tag{1.6}$$

The upper limits of W_n can be described as follows:

Theorem 1.2 *Assume* (1.1), (1.2) *and* (1.3). *For any function* $f \uparrow \infty$, $\mathbb{P}^*$*-almost surely,*

$$\limsup_{n\to\infty} \frac{\sqrt{n}\, W_n}{f(n)} = \begin{cases} 0 \\ \infty \end{cases} \iff \int^\infty \frac{dt}{t f(t)} \begin{cases} < \infty \\ = \infty \end{cases}. \tag{1.7}$$

Concerning the lower limits of W_n, we confirm (1.6) under a stronger integrability assumption: There exists some small constant $\varepsilon_0 > 0$ such that

$$\mathbb{E}\left[\eta^{1+\varepsilon_0} + \int e^{-x}|x|^{2+\varepsilon_0} \mathcal{L}(dx)\right] < \infty. \tag{1.8}$$

It is easy to see that the condition (1.8) is stronger than (1.2) and (1.3).

Proposition 1.3 *Assume* (1.1) *and* (1.8). *We have*

$$\liminf_{n\to\infty} \sqrt{n}\, W_n = \sqrt{\frac{2}{\pi\sigma^2}}\, D_\infty, \qquad \mathbb{P}^*\text{-a.s.}$$

Combining Theorems 1.1 and 1.2, we can roughly say that the main contribution to the upper limits of W_n comes from the term $e^{-\mathbb{M}_n}$. According to Madaule [25], and Aïdékon, Berestycki, Brunet and Shi [3], Arguin, Bovier and Kistler [6] (for the branching brownian motion), the branching random walk seen from the minimal position converges in law to some point process, in particular, $W_n e^{\mathbb{M}_n}$ converges in law as $n \to \infty$, but we are not able to determine the almost sure fluctuations of $W_n e^{\mathbb{M}_n}$.

The whole paper uses essentially the techniques developed by Aïdékon and Shi [5]. To show Theorems 1.1 and 1.2, we firstly remark that both two theorems share the same integral test and that since $W_n \geq e^{-\mathbb{M}_n}$, it is enough to prove the convergence part in the integral test (1.7) and the divergence part in (1.4). The convergence part in (1.7) will follow from an application of Doob's maximal inequality to a certain martingale. To prove the divergence part in (1.4), we shall use the arguments in Aïdékon and Shi [5] (the proof of their Lemma 6.3) to estimate a second moment, then apply Borel-Cantelli's lemma. We can also directly prove Theorem 1.2 without the use of the divergence part of (1.4). Finally, the proof of Proposition 1.3 relies on a result (Lemma 4.1) which is also implicitly contained in Aïdékon and Shi [5] (by following the proof of their Proposition 4.1).

The rest of this paper is organized as follows: In Section 2, we recall some known results on the branching random walk (many-to-one formula, change of measure) and on a real-valued random walk. In Section 3, we prove Theorems 1.1 and 1.2, whereas the proof of Proposition 1.3 will be given in Section 4.

Throughout this paper, $f(n) \sim g(n)$ as $n \to \infty$ means that $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$ and $(c_i, 1 \leq i \leq 36)$ denote some positive constants.

2 Preliminaries

2.1 Many-to-one formula for the branching random walk

In this subsection, we recall some change of measure formulas in the branching random walk, for the details we refer to [9, 13, 24, 5, 27] and the references therein.

At first let us fix some notations which will be used throughout this paper: For $|u| = n$, we write $[\varnothing, u] \equiv \{u_0 := \varnothing, u_1, ..., u_{n-1}, u_n = u\}$ the shortest path from the root $\varnothing$ to u such that $|u_i| = i$ for any $0 \leq i \leq n$. For any $u, v \in \mathbb{T}$, we use the partial order $u < v$ if u is an ancestor of v and $u \leq v$ if $u < v$ or $u = v$. We also denote by $\overleftarrow{v}$ the parent of v.

Under (1.1), there exists a centered real-valued random walk $\{S_n, n \geq 0\}$ such that for any $n \geq 1$ and any measurable function $f : \mathbb{R}^n \to \mathbb{R}_+$,

$$\mathbb{E}\Big[\sum_{|u|=n} e^{-V(u)} f(V(u_1), ..., V(u_n))\Big] = \mathbb{E}\left[f(S_1, ..., S_n)\right]. \tag{2.1}$$

Moreover under (1.2), $\sigma^2 = \mathrm{Var}(S_1) = \mathbb{E}\Big[\sum_{|u|=1}(V(u))^2 e^{-V(u)}\Big] \in (0, \infty)$.

The renewal function $R(x)$ related to the random walk S is defined as follows:

$$R(x) := \sum_{k=0}^{\infty} \mathbb{P}\left(S_k \geq -x,\, S_k < \min_{0\leq j\leq k-1} S_j\right), \qquad x \geq 0, \tag{2.2}$$

and $R(x) = 0$ if $x < 0$. Moreover,

$$\lim_{x\to\infty} \frac{R(x)}{x} = c_R, \tag{2.3}$$

with some positive constant c_R (see Feller [16], pp.612).

For $\alpha \ge 0$, we define as in Aïdékon and Shi [5] two truncated processes: For any $n \ge 0$,

$$W_n^{(\alpha)} := \sum_{|u|=n} e^{-V(u)} 1_{(\underline{V}(u) \ge -\alpha)}, \tag{2.4}$$

$$D_n^{(\alpha)} := \sum_{|u|=n} R_\alpha(V(u)) e^{-V(u)} 1_{(\underline{V}(u) \ge -\alpha))}, \tag{2.5}$$

where $\underline{V}(u) := \min_{\varnothing \le v \le u} V(v)$, $R_\alpha(x) := R(\alpha + x)$ and R is the renewal function defined in (2.2).

Denote by $(\mathcal{F}_n, n \ge 0)$ the natural filtration of the branching random walk. If the branching random walk starts from $V(\varnothing) = x$, then we denote its law by $\mathbb{P}_x$ (with $\mathbb{P} = \mathbb{P}_0$). According to Biggins and Kyprianou [9], $(D_n^{(\alpha)}, n \ge 0)$ is a $(\mathbb{P}_x, (\mathcal{F}_n))$-martingale and on some enlarged probability space (more precisely on the space of marked trees enlarged by an infinite ray $(\xi_n, n \ge 0)$, called spine), we may construct a family of probabilities $(\mathbb{Q}_x^{(\alpha)}, x \ge -\alpha)$ such that for any $x \ge -\alpha$, the following statements (i), (ii) and (iii) hold:

(i) For all $n \ge 1$,

$$\frac{d\mathbb{Q}_x^{(\alpha)}}{d\mathbb{P}_x}\Big|_{\mathcal{F}_n} = \frac{D_n^{(\alpha)}}{D_0^{(\alpha)}}, \tag{2.6}$$

$$\mathbb{Q}_x^{(\alpha)}\left(\xi_n = u \middle| \mathcal{F}_n\right) = \frac{1}{D_n^{(\alpha)}} R_\alpha(V(u)) e^{-V(u)} 1_{(\underline{V}(u) \ge -\alpha)}, \qquad \forall |u| = n. \tag{2.7}$$

(ii) Under $\mathbb{Q}_x^{(\alpha)}$, the process $\{V(\xi_n), n \ge 0\}$ along the spine $(\xi_n)_{n \ge 0}$, is distributed as the random walk $(S_n, n \ge 0)$ under $\mathbb{P}$ conditioned to stay in $[-\alpha, \infty)$. Moreover for any $n \ge 1, x \ge -\alpha$ and $f : \mathbb{R}^n \to \mathbb{R}_+$,

$$\mathbb{E}_{\mathbb{Q}_x^{(\alpha)}}\Big[f(V(\xi_1), ..., V(\xi_n))\Big] = \frac{1}{R_\alpha(x)} \mathbb{E}_x\Big[f(S_1, ..., S_n) R_\alpha(S_n) 1_{(\underline{S}_n \ge -\alpha)}\Big]. \tag{2.8}$$

(iii) Let $\mathcal{G}_n := \sigma\{u, V(u) : \overleftarrow{u} \in \{\xi_k, 0 \le k < n\}\}$, $n \ge 0$. Under $\mathbb{Q}_x^{(\alpha)}$ and conditioned on $\mathcal{G}_\infty$, for all $u \notin \{\xi_k, k \ge 0\}$ but $\overleftarrow{u} \in \{\xi_k, k \ge 0\}$ the induced branching random walk $(V(uv), |v| \ge 0)$ are independent and are distributed as $\mathbb{P}_{V(u)}$, where $\{uv, |v| \ge 0\}$ denotes the subtree of $\mathbb{T}$ rooted at u.

Let us mention that as a consequence of (i), the following many-to-one formula holds: For any $n \ge 1, x \ge -\alpha$ and $f : \mathbb{R}^n \to \mathbb{R}_+$,

$$\mathbb{E}_x\Big[\sum_{|u|=n} e^{-V(u)} R_\alpha(V(u)) f(V(u_1), ..., V(u_n)) 1_{(\underline{V}(u) \ge -\alpha)}\Big] = R_\alpha(x) e^{-x}\, \mathbb{E}_{\mathbb{Q}_x^{(\alpha)}}\Big[f(V(\xi_1), ..., V(\xi_n))\Big]. \tag{2.9}$$

2.2 Estimates on a centered real-valued random walk

We collect here some estimates on a real-valued random walk $\{S_k, k \ge 0\}$, centered and with finite variance $\sigma^2 > 0$. Let $\underline{S}_n := \min_{0 \le i \le n} S_i, \forall\, n \ge 0$. Recall (2.2) for the renewal function $R(\cdot)$.

Fact 2.1 *There exists some constant $c_1 > 0$ such that for any $x \ge 0$,*

$$\mathbb{P}_x\Big(\underline{S}_n \ge 0\Big) \le c_1 (1+x) n^{-1/2}, \qquad \forall\, n \ge 1, \tag{2.10}$$

$$\mathbb{P}_x\big(\underline{S}_{n-1} > S_n \ge 0\big) \le c_1 (1+x) R(x) n^{-3/2}, \qquad \forall\, n \ge 1, \tag{2.11}$$

$$\mathbb{P}_x\Big(\underline{S}_n \ge 0\Big) \sim \theta\, R(x) n^{-1/2}, \qquad \textit{as } n \to \infty, \tag{2.12}$$

with $\theta = \frac{1}{c_R}\sqrt{\frac{2}{\pi\sigma^2}}$. *Moreover there is* $c_2 > 0$ *such that for any* $b \ge a \ge 0, x \ge 0, n \ge 1$,

$$\mathbb{P}_x\Big(S_n \in [a,b], \underline{S}_n \ge 0\Big) \le c_2(1+x)(1+b-a)(1+b)n^{-3/2}, \tag{2.13}$$

For any $0 < r < 1$, *there exists some* $c_3 = c_3(r) > 0$ *such that for all* $b \ge a \ge 0, x, y \ge 0, n \ge 1$,

$$\mathbb{P}_x\left(S_n \in [y+a, y+b],\, \underline{S}_n \ge 0,\, \min_{rn \le j \le n} S_j \ge y\right) \le c_3\,(1+x)(1+b-a)(1+b)n^{-3/2}. \tag{2.14}$$

See Feller ([16], Theorem 1a, pp.415) for (2.10), Aïdékon and Jaffuel ([4], equation (2.8)) for (2.11), Aïdékon and Shi [5] for (2.13) and (2.14), and Kozlov [22] and Lemma 2.1 in [5] for (2.12) with the identification of the constant $\theta = \frac{1}{c_R}\sqrt{\frac{2}{\pi\sigma^2}}$.

We end this section by an estimate on the stability on x in the convergence (2.12).

Lemma 2.2 *Let* S *be a centered random walk with positive variance. There exists a constant* $c_4 > 0$ *such that for all* $n \ge 1$ *and* $x \ge 0$,

$$\Big|\frac{\mathbb{P}_x(\underline{S}_n \ge 0)}{R(x)\mathbb{P}(\underline{S}_n \ge 0)} - 1\Big| \le c_4 \frac{1+x}{\sqrt{n}}.$$

Proof of Lemma 2.2. Denote in this proof by $\varrho(n) := \mathbb{P}(\underline{S}_n \ge 0)$ for $n \ge 0$. Let $x \ge 0$. By considering the first $k \in [0, n]$ such that $S_k = \underline{S}_n$, we get that

$$\begin{aligned}
\mathbb{P}_x(\underline{S}_n \ge 0) &= \mathbb{P}_x(\underline{S}_n \ge x) + \sum_{k=1}^{n} \mathbb{P}_x\Big(\underline{S}_{k-1} > S_k \ge 0,\, \min_{k<j\le n} S_j \ge S_k\Big) \\
&= \varrho(n) + \sum_{k=1}^{n} \mathbb{P}_x\Big(\underline{S}_{k-1} > S_k \ge 0\Big)\varrho(n-k),
\end{aligned}$$

by the Markov property at k. Note that $R(x) = 1 + \sum_{k=1}^{\infty} \mathbb{P}_x\big(\underline{S}_{k-1} > S_k \ge 0\big)$. It follows that

$$\mathbb{P}_x(\underline{S}_n \ge 0) \le R(x)\varrho(n) + \sum_{k=1}^{n} \mathbb{P}_x\Big(\underline{S}_{k-1} > S_k \ge 0\Big)[\varrho(n-k) - \varrho(n)], \tag{2.15}$$

and

$$\mathbb{P}_x(\underline{S}_n \ge 0) \ge R(x)\varrho(n) - \sum_{k=n+1}^{\infty} \mathbb{P}_x\Big(\underline{S}_{k-1} > S_k \ge 0\Big)\varrho(n). \tag{2.16}$$

Denote respectively by $I_{(2.15)}$ and $I_{(2.16)}$ the sum $\sum_{k=1}^{n}$ in (2.15) and the sum $\sum_{k=n+1}^{\infty}$ in (2.16). Let $T^- := \inf\{j \ge 1 : S_j < 0\}$. By the local limit theorem (Eppel [15], see also [28], equation (22)), if the distribution of S_1 is non-lattice, then

$$\mathbb{P}\Big(T^- = k\Big) \sim \frac{C_-}{k^{3/2}}, \qquad k \to \infty, \tag{2.17}$$

with some positive constant C_-. Moreover Eppel [15] mentioned that a modification of (2.17) holds in the lattice distribution case. Then there exists some constant $c_5 > 0$ such that for all $k \ge 1$,

$$\mathbb{P}\Big(T^- = k\Big) \le \frac{c_5}{k^{3/2}}. \tag{2.18}$$

It follows that for any $k \le n$, $\varrho(n-k) - \varrho(n) = \mathbb{P}\Big(n-k < T^- \le n\Big) \le c_5 \sum_{i=n-k+1}^{n} i^{-3/2}$. Then by (2.11),

$$I_{(2.15)} \le c_6(1+x)R(x) \sum_{k=1}^{n} k^{-3/2} \sum_{i=n-k+1}^{n} i^{-3/2}.$$

Elementary computations show that $\sum_{k=1}^{n/2} k^{-3/2} \sum_{i=n-k+1}^{n} i^{-3/2} \le \sum_{k=1}^{n/2} k^{-3/2} \times k(\frac{n}{2})^{-3/2} = O(\frac{1}{n})$ and $\sum_{k=n/2}^{n} k^{-3/2} \sum_{i=n-k+1}^{n} i^{-3/2} \le (\frac{n}{2})^{-3/2} \sum_{i=1}^{n} i^{-3/2} \times i = O(\frac{1}{n})$. Hence $I_{(2.15)} \le c_7(1+x)R(x)\frac{1}{n} \le c_8(1+x)R(x)\frac{1}{\sqrt{n}}\varrho(n)$ by (2.12).

Finally again by (2.11), we get that $I_{(2.16)} \le c_9(1+x)R(x)\frac{1}{\sqrt{n}}\varrho(n)$. Then the Lemma follows from (2.15) and (2.16). $\Box$

3 Proofs of Theorems 1.1 and 1.2

In view of the inequality: $W_n \ge e^{-\mathbb{M}_n}$, the convergence part of the integral test (1.7) yields that of (1.4), whereas the divergence part of the integral test (1.4) implies that of (1.7). We only need to show the convergence part in (1.7) and the divergence part in (1.4).

3.1 Proof of the convergence part in Theorem 1.2:

Lemma 3.1 *Assume* (1.1). *For any $\alpha \ge 0$, there exists some constant $c_{10} = c_{10}(\alpha) > 0$ such that for any $1 < n \le m$ and $\lambda > 0$, we have*

$$\mathbb{P}\Big(\max_{n \le k \le m} \sqrt{k} W_k^{(\alpha)} > \lambda\Big) \le c_{10} \frac{\log n}{\sqrt{n}} + c_{10} \frac{1}{\lambda} \sqrt{\frac{m}{n}}.$$

Proof of Lemma 3.1. For $n \le k \le m+1$, define

$$\widetilde{W}_k^{(\alpha,n)} := \sum_{|u|=k} e^{-V(u)} 1_{(\underline{V}(u_n) \ge -\alpha)},$$

where as before $\underline{V}(u_n) := \min_{1 \le j \le n} V(u_j)$ and u_n is the ancestor of u at n-th generation. Then $\widetilde{W}_n^{(\alpha,n)} = W_n^{(\alpha)}$.

For $k \in [n,m]$, $\widetilde{W}_{k+1}^{(\alpha,n)} = \sum_{|v|=k} 1_{(\underline{V}(v_n) \ge -\alpha)} \sum_{u: \overleftarrow{u} = v} e^{-V(u)}$. The branching property implies that $\mathbb{E}\Big(\widetilde{W}_{k+1}^{(\alpha,n)} | \mathcal{F}_k\Big) = \widetilde{W}_k^{(\alpha,n)}$ for $k \in [n,m]$. By Doob's maximal inequality,

$$\mathbb{P}\Big(\max_{n \le k \le m} \sqrt{k} \widetilde{W}_k^{(\alpha,n)} \ge \lambda\Big) \le \frac{\sqrt{m}}{\lambda} \mathbb{E}(\widetilde{W}_m^{(\alpha,n)}) = \frac{\sqrt{m}}{\lambda} \mathbb{E}(W_n^{(\alpha)}).$$

By the many-to-one formula (2.1) and the random walk estimate (2.10),

$$\mathbb{E}(W_n^{(\alpha)}) = \mathbb{P}\Big(\underline{S}_n \ge -\alpha\Big) \le \frac{c_{11}}{\sqrt{n}},$$

with $c_{11} := c_1(1+\alpha)$. It follows that

$$\mathbb{P}\Big(\max_{n \le k \le m} \sqrt{k} \widetilde{W}_k^{(\alpha,n)} \ge \lambda\Big) \le \frac{c_{11}}{\lambda} \sqrt{\frac{m}{n}}.$$

Comparing $\widetilde{W}_k^{(\alpha,n)}$ and $W_k^{(\alpha)}$, we get that

$$\mathbb{P}\Big(\max_{n\le k\le m}\sqrt{k}W_k^{(\alpha)}>\lambda\Big)\le \mathbb{P}\Big(\min_{n\le k\le m}\min_{|u|=k}V(u)<-\alpha\Big)+\frac{c_{11}}{\lambda}\sqrt{\frac{m}{n}}.$$

The proof of the Lemma will be finished if we can show that for all $n\ge 2$,

$$\mathbb{P}\Big(\min_{|u|\ge n}V(u)<-\alpha\Big)\le c_{10}\frac{\log n}{\sqrt{n}}. \tag{3.1}$$

To this end, let us apply the following known result (see e.g. [27]):

$$\mathbb{P}\Big(\inf_{u\in\mathbb{T}}V(u)<-x\Big)\le e^{-x},\qquad \forall\, x\ge 0.$$

Then for all $n\ge 2$,

$$\begin{aligned}
&\mathbb{P}\Big(\min_{k\ge n}\min_{|u|=k}V(u)<-\alpha\Big)\\
\le\ & \mathbb{P}\Big(\inf_{u\in\mathbb{T}}V(u)<-\log n\Big)+\mathbb{P}\Big(\min_{k\ge n}\min_{|u|=k}V(u)<-\alpha,\inf_{v\in\mathbb{T}}V(v)\ge-\log n\Big)\\
\le\ & \frac{1}{n}+\sum_{k=n}^{\infty}\mathbb{E}\Big[\sum_{|u|=k}1_{(V(u)<-\alpha,V(u_n)\ge-\alpha,\dots,V(u_{k-1})\ge-\alpha,\,\underline{V}(u)\ge-\log n)}\Big]\\
=\ & \frac{1}{n}+\sum_{k=n}^{\infty}\mathbb{E}\Big[e^{S_k}1_{(S_k<-\alpha,S_n\ge-\alpha,\dots,S_{k-1}\ge-\alpha,\,\underline{S}_k\ge-\log n)}\Big]\\
\le\ & \frac{1}{n}+e^{-\alpha}\mathbb{P}\Big(\underline{S}_n\ge-\log n\Big),
\end{aligned}$$

where the above equality is due to the many-to-one formula (2.1). Using (2.10) to bound the above probability term, we get (3.1) and the Lemma. □

Proof of the convergence part in Theorem 1.2: Let f be nondecreasing such that $\int^\infty \frac{dt}{tf(t)}<\infty$. Let $n_j:=2^j$ for large $j\ge j_0$. Then $\sum_{j=j_0}^\infty\frac{1}{f(n_j)}<\infty$. By using Lemma 3.1,

$$\mathbb{P}\Big(\max_{n_j\le k\le n_{j+1}}\sqrt{k}W_k^{(\alpha)}>f(n_j)\Big)\le c_{10}\frac{\log n_j}{\sqrt{n_j}}+c_{10}\,\frac{\sqrt{2}}{f(n_j)},$$

whose sum on j converges. The Borel-Cantelli lemma implies that $\mathbb{P}$-a.s. for all large k, $\sqrt{k}W_k^{(\alpha)}\le f(k)$. Replacing $f(k)$ by $\varepsilon f(k)$ with an arbitrary constant $\varepsilon>0$, we get that

$$\limsup_{k\to\infty}\frac{\sqrt{k}W_k^{(\alpha)}}{f(k)}=0,\qquad \mathbb{P}\text{-a.s.},$$

for any $\alpha\ge 0$. By considering a countable $\alpha\to\infty$ (for instance α integer) and by using the fact that $W_k^{(\alpha)}=W_k$ on the set $\{\inf_{u\in\mathbb{T}}V(u)\ge-\alpha\}$, we get the convergence part. □.

3.2 Proof of the divergence part in Theorem 1.1:

The following lemma is a slight modification of Aïdékon and Shi [5]'s Lemma 6.3:

Lemma 3.2 ([5]) *There exist some constants $K > 0$ and $c_{12} = c_{12}(K) > 0$ such that for all $n \geq 2, 0 \leq \lambda \leq \frac{1}{3}\log n$,*

$$c_{12}\, e^{-\lambda} \leq \mathbb{P}\Big(\bigcup_{k=n+1}^{2n} \big(E_k^{(n,\lambda)} \cap F_k^{(n,\lambda)}\big) \neq \emptyset \Big) \leq \frac{1}{c_{12}} e^{-\lambda}, \tag{3.2}$$

where for $n < k \leq 2n$,

$$\begin{aligned} E_k^{(n,\lambda)} &:= \Big\{ u : |u| = k, \frac{1}{2}\log n - \lambda \leq V(u) \leq \frac{1}{2}\log n - \lambda + K,\, V(u_i) \geq a_i^{(n,\lambda)}, \forall\, 0 \leq i \leq k \Big\}, \\ F_k^{(n,\lambda)} &:= \Big\{ u : |u| = k, \sum_{v \in \Upsilon(u_{i+1})} (1 + (V(v) - a_i^{(n,\lambda)})_+) e^{-(V(v) - a_i^{(n,\lambda)})} \leq K\, e^{-b_i^{(k,n)}},\, \forall\, 0 \leq i < k \Big\}, \end{aligned}$$

where for $u \in \mathbb{T}\backslash\{\varnothing\}$, $\Upsilon(u) := \{v : v \neq u, \overleftarrow{v} = \overleftarrow{u}\}$ denotes the set of brothers of u, $x_+ := \max(x, 0)$,

$$a_i^{(n,\lambda)} := \Big(\frac{1}{2}\log n - \lambda\Big)\, 1_{(\frac{n}{2} < i \leq 2n)}, \qquad 0 \leq i \leq 2n,$$

and for $n < k \leq 2n$,

$$b_i^{(k,n)} := i^{1/12} 1_{(0 \leq i \leq \frac{n}{2})} + (k-i)^{1/12}\, 1_{(\frac{n}{2} < i \leq k)}, \qquad 0 \leq i \leq k.$$

Proof of Lemma 3.2. The proof of the lower bound in (3.2) [by the second moment method] goes in the same way as that of Lemma 6.3 in Aïdékon and Shi [5] [We also keep their notations], by replacing $\frac{1}{2}\log n$ in their proof by $\frac{1}{2}\log n - \lambda$. Moreover, a similar computation of the second moment will be given in the proof of Lemma 3.3. Then we omit the details.

The upper bound in (3.2) is a simple consequence of the many-to-one formula: Defining $s := \frac{1}{2}\log n - \lambda$, we have that

$$\begin{aligned} \mathbb{P}\Big(\bigcup_{k=n+1}^{2n} E_k^{(n,\lambda)} \neq \emptyset \Big) &\leq \sum_{k=n+1}^{2n} \mathbb{E}\Big[\sum_{|u|=k} 1_{(s \leq V(u) \leq s+K, V(u_i) \geq a_i^{(n,\lambda)}, \forall i \leq k)} \Big] \\ &= \sum_{k=n+1}^{2n} \mathbb{E}\Big[e^{S_k} 1_{(s \leq S_k \leq s+K, S_i \geq a_i^{(n,\lambda)}, \forall i \leq k)} \Big] \\ &\leq \sum_{k=n+1}^{2n} e^{s+K}\, \mathbb{P}\Big(s \leq S_k \leq s+K, S_i \geq a_i^{(n,\lambda)}, \forall i \leq k \Big). \end{aligned}$$

By (2.14), $\mathbb{P}\big(s \leq S_k \leq s+K, S_i \geq a_i^{(n,\lambda)}, \forall i \leq k\big) \leq c_{13} n^{-3/2}$ for all $n < k \leq 2n$. Hence $\mathbb{P}\big(\bigcup_{k=n+1}^{2n} E_k^{(n,\lambda)} \neq \emptyset\big) \leq c_{13} e^{-\lambda+K}$ proving the upper bound in (3.2). $\Box$

Using the notations in Lemma 3.2 with the constant K, we define for $n \geq 2$ and $0 \leq \lambda \leq \frac{1}{3}\log n$,

$$A(n,\lambda) := \Big\{ \cup_{k=n+1}^{2n} \big(E_k^{(n,\lambda)} \cap F_k^{(n,\lambda)}\big) \neq \emptyset \Big\}. \tag{3.3}$$

The following estimate will be useful in the application of Borel-Cantelli's lemma:

Lemma 3.3 *There exists some constant $c_{14} > 0$ such that for any $n \ge 2, 0 \le \lambda \le \frac{1}{3}\log n$ and $m \ge 4n, 0 \le \mu \le \frac{1}{3}\log m$,*

$$\mathbb{P}\Big(A(n,\lambda)\cap A(m,\mu)\Big) \le c_{14}\, e^{-\lambda-\mu} + c_{14}\, e^{-\mu}\frac{\log n}{\sqrt{n}}.$$

Proof of Lemma 3.3. As we mentioned before, the arguments that we use are very close to the computation of the second moment in the proof of Lemma 6.3 in [5]. The introduction of the events $F_k^{(n,\lambda)}$ in $A(n,\lambda)$, sometimes called a truncation argument, is necessary to control the second moment: the event $F_k^{(n,\lambda)}$ keeps the path $(V(u_i), 0 \le i \le k)$ of a particle u in $E_k^{(n,\lambda)}$ to stay far away from $(a_i^{(n,\lambda)}, 0 \le i \le k)$, otherwise the particle u would give a too large expectation in the second moment. Such truncation argument was already introduced in Aïdékon [2].

Let us enter into the details of the proof of Lemma 3.3. Write for brevity

$$s := \frac{1}{2}\log n - \lambda, \qquad t := \frac{1}{2}\log m - \mu.$$

Similarly to (2.6) and (2.7), we may construct a new probability $\mathbb{Q}$ such that for all $n \ge 1$, $\frac{d\mathbb{Q}}{d\mathbb{P}}\big|_{\mathcal{F}_n} = W_n$, $\mathbb{Q}\big(\xi_n = u\big|\mathcal{F}_n\big) = \frac{e^{-V(u)}}{W_n}, \forall |u| = n$. Moreover under $\mathbb{Q}$, $(V(\xi_n), n \ge 0)$ is distributed as the random walk $(S_n, n \ge 0)$ defined in Section 2, and the spine decomposition similar to (iii) in Section 2 holds under $\mathbb{Q}$. We refer to [9, 13, 24, 5, 27] for details. It follows that

$$\begin{aligned}
\mathbb{P}\Big(A(n,\lambda)\cap A(m,\mu)\Big) &\le \mathbb{E}\Big[1_{A(n,\lambda)} \sum_{k=m+1}^{2m}\sum_{|u|=k} 1_{(u\in E_k^{(m,\mu)}\cap F_k^{(m,\mu)})}\Big] \\
&= \sum_{k=m+1}^{2m} \mathbb{E}_{\mathbb{Q}}\Big[1_{A(n,\lambda)} e^{V(\xi_k)} 1_{(\xi_k\in E_k^{(m,\mu)}\cap F_k^{(m,\mu)})}\Big] \\
&\le e^{t+K}\sum_{k=m+1}^{2m} \mathbb{E}_{\mathbb{Q}}\Big[A(n,\lambda),\, \xi_k \in E_k^{(m,\mu)} \cap F_k^{(m,\mu)}\Big] \\
&\le e^{t+K}\sum_{k=m+1}^{2m}\sum_{l=n+1}^{2n} \mathbb{E}_{\mathbb{Q}}\Big[\sum_{|v|=l} 1_{(v\in E_l^{(n,\lambda)}\cap F_l^{(n,\lambda)},\, \xi_k\in E_k^{(m,\mu)}\cap F_k^{(m,\mu)})}\Big] \\
&=: e^{t+K}\sum_{k=m+1}^{2m}\sum_{l=n+1}^{2n} I_{(3.4)}(k,l). \qquad (3.4)
\end{aligned}$$

For $n < l \le 2n \le \frac{m}{2} < k \le 2m$, we may decompose the sum on $|v| = l$ as follows:

$$\sum_{|v|=l} 1_{(v\in E_l^{(n,\lambda)}\cap F_l^{(n,\lambda)})} = 1_{(\xi_l\in E_l^{(n,\lambda)}\cap F_l^{(n,\lambda)})} + \sum_{p=1}^{l}\sum_{u\in\Upsilon(\xi_p)}\sum_{v\in\mathbb{T}(u), |v|_u = l-p} 1_{(v\in E_l^{(n,\lambda)}\cap F_l^{(n,\lambda)})},$$

where $\mathbb{T}(u)$ denotes the subtree of $\mathbb{T}$ rooted at u and $|v|_u = |v| - |u|$ the relative generation of $v \in \mathbb{T}(u)$. Then

$$\begin{aligned} & I_{(3.4)}(k,l) \\ = \;& \mathbb{Q}\Big(\xi_l \in E_l^{(n,\lambda)} \cap F_l^{(n,\lambda)}, \xi_k \in E_k^{(m,\mu)} \cap F_k^{(m,\mu)}\Big) + \sum_{p=1}^{l} \mathbb{E}_{\mathbb{Q}}\Big[1_{(\xi_k \in E_k^{(m,\mu)} \cap F_k^{(m,\mu)})} \sum_{u \in \Upsilon(\xi_p)} f_{k,l,p}(V(u))\Big] \\ =: \;& I_{(3.5)}(k,l) + \sum_{p=1}^{l} J_{(3.5)}(k,l,p), \end{aligned} \tag{3.5}$$

with

$$f_{k,l,p}(x) := \mathbb{E}_{\mathbb{Q}}\Big[\sum_{v \in \mathbb{T}(u), |v|_u = l-p} 1_{(v \in E_l^{(n,\lambda)} \cap F_l^{(n,\lambda)})} \big| V(u) = x\Big], \qquad x \in \mathbb{R}.$$

In what follows, we shall at first estimate $J_{(3.5)}(k,l,p)$ then $I_{(3.5)}(k,l)$. By the branching property at u and by removing the event $F_l^{(n,\lambda)}$ from the indicator function in $f_{k,l,p}(r)$, we get that

$$\begin{aligned} f_{k,l,p}(x) &\le \mathbb{E}_x\Big[\sum_{|v|=l-p} 1_{(s \le V(v) \le s+K, V(v_i) \ge a_{i+p}^{(n,\lambda)}, \forall 0 \le i \le l-p)}\Big] \\ &= e^{-x}\,\mathbb{E}_x\Big[e^{S_{l-p}} 1_{(s \le S_{l-p} \le s+K, S_i \ge a_{i+p}^{(n,\lambda)}, \forall 0 \le i \le l-p)}\Big] \\ &\le e^{-x+s+K}\,\mathbb{P}_x\Big(s \le S_{l-p} \le s+K, S_i \ge a_{i+p}^{(n,\lambda)}, \forall 0 \le i \le l-p\Big), \end{aligned} \tag{3.6}$$

where to get the above equality, we applied an obvious modification of (2.1) for $\mathbb{E}_x$ instead of $\mathbb{E}$.

Let us denote by $(3.6)_{k,l,p}$ the probability term in (3.6). To estimate $(3.6)_{k,l,p}$, we distinguish as in [5] two cases: $p \le \frac{n}{2}$ and $\frac{n}{2} < p \le l$. Recall that $n < l \le 2n \le \frac{m}{2} < k \le 2m$. If $p \le \frac{n}{2}$,

$$(3.6)_{k,l,p} \le 1_{(x \ge 0)} c_{15} \frac{1+x}{(l-p)^{3/2}},$$

by using (2.14). Then for $1 \le p \le \frac{n}{2}$,

$$f_{k,l,p}(x) \le c_{15} 1_{(x \ge 0)} e^{s+K-x}(1+x)(l-p)^{-3/2}.$$

It follows that for all $n < l \le 2n, m < k \le 2m$,

$$\begin{aligned} \sum_{1 \le p \le n/2} J_{(3.5)}(k,l,p) &\le \sum_{p=1}^{n/2} \mathbb{E}_{\mathbb{Q}}\Big[1_{(\xi_k \in E_k^{(m,\mu)} \cap F_k^{(m,\mu)})} \sum_{u \in \Upsilon(\xi_p)} c_{15} 1_{(V(u) \ge 0)} e^{s+K-V(u)} \frac{1+V(u)}{(l-p)^{3/2}}\Big] \\ &\le c_{16}\, e^s n^{-3/2} \sum_{p=1}^{n/2} \mathbb{E}_{\mathbb{Q}}\Big[1_{(\xi_k \in E_k^{(m,\mu)} \cap F_k^{(m,\mu)})} \sum_{u \in \Upsilon(\xi_p)} 1_{(V(u) \ge 0)} e^{-V(u)}(1+V(u))\Big] \\ &\le c_{16}\, K\, e^s n^{-3/2} \sum_{p=1}^{n/2} \mathbb{E}_{\mathbb{Q}}\Big[1_{(\xi_k \in E_k^{(m,\mu)} \cap F_k^{(m,\mu)})} e^{-(p-1)^{1/12}}\Big], \end{aligned}$$

where the last inequality is due to the definition of $\xi_k \in F_k^{(m,\mu)}$ [noticing that $a_p^{(m,\mu)} = 0$ and $b_p^{(k,m)} = p^{1/12}$ for all $p \le n/2 < m/2$]. Then we get that

$$\sum_{1\le p\le n/2} J_{(3.5)}(k,l,p) \le c_{17} e^s n^{-3/2}\, \mathbb{Q}\Big(\xi_k \in E_k^{(m,\mu)}\Big) \le c_{18} e^s n^{-3/2}\, m^{-3/2}, \tag{3.7}$$

since $\mathbb{Q}\Big(\xi_k \in E_k^{(m,\mu)}\Big) = \mathbb{P}\Big(t \le S_k \le t+K, S_i \ge a_i^{(m,\mu)}, \forall\, 0 \le i \le k\Big) \le c_{19} m^{-3/2}$ for all $m < k \le 2m$, by using (2.14).

Now considering $\frac{n}{2} < p \le l$, $a_{i+p}^{(n,\lambda)} = s$ for any $0 \le i \le l-p$, hence

$$(3.6)_{k,l,p} = 1_{(x\ge s)} \mathbb{P}_x\Big(s \le S_{l-p} \le s+K, \underline{S}_{l-p} \ge s\Big) \le 1_{(x \ge s)} c_2 (1+K)^2 \, \frac{1+x-s}{(1+l-p)^{3/2}},$$

by (2.13). It follows that

$$\sum_{\frac{n}{2}<p\le l} J_{(3.5)}(k,l,p) \quad \le \quad \sum_{\frac{n}{2}<p\le l} \mathbb{E}_{\mathbb{Q}}\Big[1_{(\xi_k \in E_k^{(m,\mu)} \cap F_k^{(m,\mu)})} \sum_{u\in\Upsilon(\xi_p)} c_2(1+K)^2 1_{(V(u)\ge s)} e^{s+K-V(u)} \frac{1+V(u)-s}{(1+l-p)^{3/2}}\Big].$$

By the definition of $\xi_k \in F_k^{(m,\mu)}$, for all $p \le l \le 2n \le \frac{m}{2}$, we have that

$$\sum_{u\in\Upsilon(\xi_p)} 1_{(V(u)\ge s)} e^{-V(u)} (1+V(u)-s) \le \sum_{u\in\Upsilon(\xi_p)} 1_{(V(u)\ge 0)} e^{-V(u)}(1+V(u)) \le K e^{-(p-1)^{1/12}}.$$

Then

$$\sum_{\frac{n}{2}\le p\le l} J_{(3.5)}(k,l,p) \le c_{20} e^s\, e^{-n^{1/13}} \mathbb{Q}(\xi_k \in E_k^{(m,\mu)}) \le c_{21} e^{s-n^{1/13}} m^{-3/2}. \tag{3.8}$$

Combining (3.7) and (3.8), we get that

$$\sum_{1\le p\le l} J_{(3.5)}(k,l,p) \le (c_{18} n^{-3/2} + c_{21} e^{-n^{1/13}}) e^s\, m^{-3/2}. \tag{3.9}$$

It remains to estimate $I_{(3.5)}(k,l)$ for $n < l \le 2n$ and $m < k \le 2m$ [in particular $l < k$]. We have

$$\begin{aligned} & I_{(3.5)}(k,l) \\ \le\ & \mathbb{Q}\Big(\xi_l \in E_l^{(n,\lambda)}, \xi_k \in E_k^{(m,\mu)}\Big) \\ =\ & \mathbb{P}\Big(s \le S_l \le s+K, S_i \ge a_i^{(n,\lambda)}, \forall\, i \le l,\, t \le S_k \le t+K, S_j \ge a_j^{(m,\mu)}, \forall\, j \le k\Big). \end{aligned} \tag{3.10}$$

Let us denote by $(3.10)_{k,l}$ the probability term in (3.10). Using the Markov property at l, we get that

$$\begin{aligned} (3.10)_{k,l} &= \mathbb{E}\Big[1_{(s\le S_l\le s+K, S_i \ge a_i^{(n,\lambda)}, \forall 0\le i\le l)} \mathbb{P}_{S_l}\Big(t \le S_{k-l} \le t+K, S_j \ge a_{j+l}^{(m,\mu)}, \forall\, 0 \le j \le k-l\Big)\Big] \\ &\le \frac{c_{21}}{(k-l)^{3/2}} \mathbb{E}\Big[1_{(s\le S_l\le s+K, S_i \ge a_i^{(n,\lambda)}, \forall 0\le i\le l)} (1+S_l)\Big] \qquad \text{(by (2.13))} \\ &\le c_{22}(1+s+K)(k-l)^{-3/2}\, l^{-3/2}. \end{aligned}$$

Based on the above estimate and (3.9), we deduce from (3.4) and (3.5) that

$$\begin{aligned}
& \mathbb{P}\Big(A(n,\lambda)\cap A(m,\mu)\Big) \\
\le \;& c_{23}\, e^{t+K} \sum_{k=m+1}^{2m}\sum_{l=n+1}^{2n} \Big((1+s+K)(k-l)^{-3/2}\, l^{-3/2} + e^s\, e^{-n^{1/13}} m^{-3/2} + e^s n^{-3/2}\, m^{-3/2}\Big) \\
\le \;& c_{24} e^{-\lambda-\mu} + c_{24} e^{-\mu}\frac{\log n}{\sqrt{n}},
\end{aligned}$$

proving the Lemma. □

Proof of the divergence part in Theorem 1.1. Let f be nondecreasing such that $\int^\infty \frac{dt}{te^{f(t)}} = \infty$. Without any loss of generality we may assume that $\sqrt{\log t} \le e^{f(t)} \le (\log t)^2$ for all large $t \ge t_0$ (see e.g. [14] for a similar justification). Denote by

$$B_x(k) := \Big\{\mathbb{M}_n + x \le \frac{1}{2}\log n - f(n+k), \text{ i.o. as } n\to\infty\Big\}, \qquad x\in\mathbb{R},\ k\ge 0.$$

Let us first prove that there exists some constant $c_{25}>0$ such that for any $x\in\mathbb{R}$ and $k\ge 0$,

$$\mathbb{P}\Big(B_x(k)\Big) \ge c_{25}. \tag{3.11}$$

To this end, we take $n_i := 2^i$ for $i\ge 1$, $\lambda_i := f(n_{i+1}+k)+x+K$ and consider the event $A_i := A(n_i,\lambda_i)$ in (3.3). There is some integer $i_0 \equiv i_0(x,k)\ge 1$ such that for all $i\ge i_0$, $0\le \lambda_i \le \frac{1}{3}\log n_i$. By Lemma 3.2,

$$c_{12}\, e^{-\lambda_i} \le \mathbb{P}(A_i) \le \frac{1}{c_{12}}\, e^{-\lambda_i}, \qquad \forall i\ge i_0.$$

Note that $\int^\infty \frac{dt}{te^{f(t+k)}} \ge \int^\infty \frac{dt}{(t+k)e^{f(t+k)}} = \infty$, and $\int_{n_{i+1}}^{n_{i+2}} \frac{dt}{te^{f(t+k)}} \le (\log 2)e^{-f(n_{i+1}+k)}$ by the monotonicity of f. Hence $\sum_i e^{-\lambda_i} = \infty$. By Lemma 3.3, we have for any $i\ge i_0$ and $j\ge i+2$,

$$\mathbb{P}\Big(A_i\cap A_j\Big) \le c_{14}\, e^{-\lambda_i-\lambda_j} + c_{14} e^{-\lambda_j}\frac{\log n_i}{\sqrt{n_i}},$$

which implies that $\sum_{i,j=i_0}^k \mathbb{P}\Big(A_i\cap A_j\Big) \le c_{14}\big(\sum_{i=i_0}^k e^{-\lambda_i}\big)^2 + 2c_{14}\big(\sum_{i=i_0}^k e^{-\lambda_i}\big)\times\big(\sum_{i=1}^\infty \frac{\log n_i}{\sqrt{n_i}}\big)$. Using the lower bound $\mathbb{P}(A_i)\ge c_{12}\, e^{-\lambda_i}$ and the fact that $\sum_{i=i_0}^k e^{-\lambda_i}\to\infty$ as $k\to\infty$, we obtain that

$$\limsup_{k\to\infty} \frac{\sum_{1\le i,j\le k}\mathbb{P}(A_i\cap A_j)}{\big[\sum_{i=1}^k \mathbb{P}(A_i)\big]^2} \le \frac{c_{14}}{c_{12}^2}.$$

By Kochen and Stone [21]'s version of the Borel-Cantelli lemma, $\mathbb{P}(A_i, \text{i.o. } i\to\infty) \ge c_{12}^2/c_{14} =: c_{25}$ which does not depend on (x,k). Observe that $\{A_i, \text{i.o. } i\to\infty\}\subset B_x(k)$, in fact, for those i such that $A_i \equiv A(n_i,\lambda_i)$ holds, by the definition (3.3), there exits some $n\in(n_i,n_{i+1}]$ such that $\mathbb{M}_n \le \frac{1}{2}\log n_i - \lambda_i + K = \frac{1}{2}\log n_i - f(n_{i+1}+k) - x \le \frac{1}{2}\log n - f(n+k) - x$. Hence we get (3.11).

We have proved that for any $x\in\mathbb{R}$ and $k\ge 0$, $\mathbb{P}(B_x(k))\ge c_{25}$. For any $k\ge 0$, the events $B_x(k)$ are non-increasing on x. Let $B_\infty(k) := \cap_{i=1}^\infty B_i(k)$ [then $B_\infty(k)$ is nothing but $\{\liminf_{n\to\infty}(\mathbb{M}_n - \frac{1}{2}\log n +$

$f(n+k)) = -\infty\}]$. By the monotone convergence, $\mathbb{P}(B_\infty(k)) \ge c_{25}$, for all $k \ge 0$. Moreover, for any $x \in \mathbb{R}$, $\mathbb{P}_x(B_\infty(k)) = \mathbb{P}(B_\infty(k)) \ge c_{25}$. On the other hand, if we denote by $Z_k := \sum_{|u|=k} 1$ the number of particles in the k-th generation, then by the branching property,

$$\mathbb{P}\Big(B_\infty(0) \,\big|\, \mathcal{F}_k\Big) = 1_{(Z_k>0)}\Big(1 - \prod_{|u|=k} (1 - \mathbb{P}_{V(u)}(B_\infty(k)))\Big) \ge 1_{(Z_k>0)}\Big(1 - (1-c_{25})^{Z_k}\Big).$$

It is well-known (cf. [7], pp.8) that $\mathcal{S} = \{\lim_{k\to\infty} Z_k = \infty\}$. Then by letting $k \to \infty$ in the above inequality, we get that

$$1_{B_\infty(0)} = \lim_{k\to\infty} \mathbb{P}\Big(B_\infty(0) \,\big|\, \mathcal{F}_k\Big) \ge 1_{\mathcal{S}}, \qquad \mathbb{P}\text{-a.s.}$$

Clearly $\mathcal{S}^c \subset B_\infty(0)^c$ by the convention on the definition of $\mathbb{M}_n$ on $\mathcal{S}^c$. Hence $\mathcal{S} = B_\infty(0)$, $\mathbb{P}$-a.s. This proves the divergence part of Theorem 1.1. $\Box$

4 Proof of Proposition 1.3

The main technical part was already done in Aïdékon and Shi [5]:

Lemma 4.1 ([5]) *Assume* (1.1) *and* (1.8). *For any fixed* $\alpha \ge 0$, *there exist some* $\delta = \delta(\varepsilon_0) > 0$ *and* $c_{26} = c_{26}(\alpha, \delta) > 0$ *such that for all* $n \ge 2$,

$$Var_{\mathbb{Q}^{(\alpha)}}\left(\frac{\sqrt{n}W_n^{(\alpha)}}{D_n^{(\alpha)}}\right) \le c_{26}\left(n^{-\delta} + \sup_{k_n^{1/3} \le x \le k_n} \Big|\frac{h_{x+\alpha}(n-k_n)}{h_\alpha(n)} - 1\Big|\right), \tag{4.1}$$

where $k_n := \lfloor n^{1/3} \rfloor$ *and* $h_x(j) := \frac{\sqrt{j}\,\mathbb{P}_x(\underline{S}_j \ge 0)}{R(x)}$ *for* $j \ge 1$, $x \ge 0$.

Proof of Lemma 4.1. The Lemma was implicitly contained in the proof of Proposition 4.1 in [5]. In fact, in their proof of the convergence that $\mathrm{Var}_{\mathbb{Q}^{(\alpha)}}\left(\frac{\sqrt{n}W_n^{(\alpha)}}{D_n^{(\alpha)}}\right) \to 0$, we choose $k_n := \lfloor n^{1/3} \rfloor$ in their definition of $E_n := E_{n,1} \cap E_{n,2} \cap E_{n,3}$ (see the equation (4.6) in [5], Section 4). We claim that for some constant $\delta_1 = \delta_1(\varepsilon_0) > 0$, there is some $c_{27} = c_{27}(\delta_1, \alpha) > 0$ such that for all $n \ge 1$,

$$\mathbb{Q}^{(\alpha)}\Big(E_n^c\Big) \le c_{27}\, n^{-\delta_1}, \tag{4.2}$$

$$\sup_{k_n^{1/3} \le x \le k_n} \mathbb{Q}^{(\alpha)}\Big(E_n^c \big| V(\xi_{k_n}) = x\Big) \le c_{27} n^{-\delta_1}. \tag{4.3}$$

In fact, according to the definition of $E_{n,1}$ in [5],

$$\mathbb{Q}^{(\alpha)}\Big(E_{n,1}^c\Big) \le \mathbb{Q}^{(\alpha)}\Big(\{V(\xi_{k_n}) > k_n\} \cup \{V(\xi_{k_n}) < k_n^{1/3}\}\Big) + \sup_{k_n^{1/3} \le x \le k_n} \mathbb{Q}_x^{(\alpha)}\Big(\cup_{i=0}^{n-k_n} \{V(\xi_i) < k_n^{1/6}\}\Big).$$

By (2.8),

$$\mathbb{Q}^{(\alpha)}\Big(V(\xi_{k_n}) < k_n^{1/3}\Big) = \frac{1}{R_\alpha(0)}\mathbb{E}\Big(1_{(\underline{S}_{k_n} \ge -\alpha, S_{k_n} < k_n^{1/3})} R_\alpha(S_{k_n})\Big) \le \frac{R_\alpha(k_n^{1/3})}{R_\alpha(0)}\,\mathbb{P}(\underline{S}_{k_n} \ge -\alpha) \le c_{28} k_n^{-1/6},$$

and

$$\mathbb{Q}^{(\alpha)}\Big(V(\xi_{k_n}) > k_n\Big) \le \mathbb{E}\Big[1_{(S_{k_n}>k_n)} R_\alpha(S_{k_n})\Big] \le \sqrt{\mathbb{P}\Big(S_{k_n} > k_n\Big)}\sqrt{\mathbb{E}[R_\alpha(S_{k_n})^2]} \le c_{29} k_n \sqrt{\mathbb{P}\Big(S_{k_n} > k_n\Big)},$$

since $R_\alpha(x) \sim c_R x$ as $x \to \infty$. The condition (1.8) ensures that $\mathbb{E}(|S_1|^{2+\varepsilon_0}) < \infty$ which in turn implies that $\mathbb{E}(|S_k|^{2+\varepsilon_0}) \le c_{30} k^{1+\varepsilon_0/2}$ for any $k \ge 1$ (see Petrov [26], pp.60). Hence

$$\mathbb{Q}^{(\alpha)}\Big(V(\xi_{k_n}) > k_n\Big) \le c_{31} k_n^{-\varepsilon_0/4}.$$

Now for $k_n^{1/3} \le x \le k_n$, let $\tau = \inf\{i \ge 0 : S_i < k_n^{1/6}\}$, then the absolute continuity (2.8) at τ reads as

$$\begin{aligned}
\mathbb{Q}_x^{(\alpha)}\Big(\cup_{i=0}^{n-k_n}\{V(\xi_i) < k_n^{1/6}\}\Big) &= \frac{1}{R_\alpha(x)}\mathbb{E}_x\Big[1_{(\tau\le n-k_n)} R_\alpha(S_\tau) 1_{(\underline{S}_\tau \ge -\alpha)}\Big] \\
&\le \frac{R_\alpha(k_n^{1/6})}{R_\alpha(x)} \le \frac{R_\alpha(k_n^{1/6})}{R_\alpha(k_n^{1/3})} \le c_{32}\, k_n^{-1/6},
\end{aligned}$$

since $x \ge k_n^{1/3}$. Assembling the above estimates yields that

$$\mathbb{Q}^{(\alpha)}\Big(E_{n,1}^c\Big) \le c_{33} k_n^{-\varepsilon_0/4},$$

[we may assume $\varepsilon_0 \le 2/3$]. Let us follow the proof of Lemma 4.7 in [5], we remark that on $E_{n,1}$, $V(\xi_i) \ge k_n^{1/6}$ for all $k_n \le i \le n$, and it was shown in [5] that

$$\mathbb{Q}_x^{(\alpha)}\Big(E_{n,1} \cap E_{n,2}^c\Big) \le \sum_{i=k_n}^{n-1} \mathbb{E}_x\Big[1_{(\eta+\widetilde{\eta} > e^{S_i/2})}\big[\eta + \frac{\widetilde{\eta}}{S_i+\alpha+1}\big]\, 1_{(S_i \ge k_n^{1/6})}\Big].$$

By the integrability assumption (1.8), since $\widetilde{\eta} = \int_0^\infty x e^{-x}\mathcal{L}(dx) \le \sqrt{\eta \int_0^\infty x^2 e^{-x}\mathcal{L}(dx)}$, it is easy to see that $\mathbb{E}(\widetilde{\eta}^p) < \infty$ for some $p > 1$. It follows that $\mathbb{Q}_x^{(\alpha)}\Big(E_{n,1} \cap E_{n,2}^c\Big) \le n\,\mathbb{E}\Big(1_{(\eta+\widetilde{\eta} > e^{k_n^{1/6}/2})}\big[\eta + \widetilde{\eta}\big]\Big) \le c_{34} n^{-10}$. Finally by (4.9) in [5], $\mathbb{Q}^{(\alpha)}(E_{n,1} \cap E_{n,2} \cap E_{n,3}^c) \le c_{35} n^{-10}$, hence we get (4.2).

From (4.2), it suffices to follow line-by-line the proof of Proposition 4.1 in [5]: In Lemma 4.4 of [5], we can get $n^{-1-\delta_1/4}$ instead of $o(\frac{1}{n})$ [by replacing in its proof ε by $n^{-\delta_1/4}$]. In their proof of Lemma 4.5, taking $\eta_1 = \frac{1}{n}$ and we arrive at

$$\mathbb{E}_{\mathbb{Q}^{(\alpha)}}\left[\Big(\sqrt{n}\frac{W_n^{(\alpha)}}{D_n^{(\alpha)}}\Big)\right]^2 \le c_{36} n^{-\delta_1/4} + (1 + O(\frac{1}{n}))\mathbb{E}_{\mathbb{Q}^{(\alpha)}}\big[\sqrt{n}\frac{W_n^{(\alpha)}}{D_n^{(\alpha)}}\big] \sup_{k_n^{1/3} \le x \le k_n} \frac{\sqrt{n}\mathbb{P}(\underline{S}_{n-k_n} \ge -\alpha - x)}{R_\alpha(x)}.$$

The Lemma follows because $\mathbb{E}_{\mathbb{Q}^{(\alpha)}}\big[\sqrt{n}\frac{W_n^{(\alpha)}}{D_n^{(\alpha)}}\big] = h_\alpha(n)$, and $h_\alpha(n) \to \theta$ when $n \to \infty$, as shown in [5]. $\Box$

Proof of Proposition 1.3. It is enough to prove that for any $\alpha \ge 0$,

$$\liminf_{n\to\infty} \sqrt{n}\frac{W_n^{(\alpha)}}{D_n^{(\alpha)}} = \theta, \qquad \mathbb{Q}^{(\alpha)}\text{-a.s.}, \tag{4.4}$$

where θ is defined in (2.12). In fact, under (1.1), (1.2) and (1.3), $D_n^{(\alpha)}$ converges in mean to $D_\infty^{(\alpha)}$ (see [27], Chapter 5, also see [9], Theorem 10.2 (i) with an extra $\log\log\log$-term). Then on $\{D_\infty^{(\alpha)} > 0\}$, $\mathbb{P}$ and $\mathbb{Q}^{(\alpha)}$ are equivalent. Moreover, as shown in [5], $\mathbb{P}$-almost surely on $\{\inf_{|u|\ge 0} V(u) \ge -\alpha\}$, $W_n^{(\alpha)} = W_n$ and $\lim_{n\to\infty} D_n^{(\alpha)} = c_R\, D_\infty$, therefore Proposition 1.3 follows easily from (4.4).

Now to prove (4.4), since $\sqrt{n}\frac{W_n^{(\alpha)}}{D_n^{(\alpha)}} \to \theta$ in probability under $\mathbb{Q}^{(\alpha)}$ ([5]), it suffices to prove that

$$\liminf_{n\to\infty} \sqrt{n}\frac{W_n^{(\alpha)}}{D_n^{(\alpha)}} \ge \theta, \qquad \mathbb{Q}^{(\alpha)}\text{-a.s.} \tag{4.5}$$

To this end, using Lemmas 4.1 and 2.2 we get some constant $\delta_2 > 0$ such that for all $n \ge n_0$,

$$\mathrm{Var}_{\mathbb{Q}^{(\alpha)}}\Big(\sqrt{n}\frac{W_n^{(\alpha)}}{D_n^{(\alpha)}}\Big) \le n^{-\delta_2}. \tag{4.6}$$

Let $n_j := j^{-3/\delta_2}$ for $j \ge j_0$ and choose an arbitrary small $\varepsilon > 0$. We are going to show that

$$\sum_{j\ge j_0} \mathbb{Q}^{(\alpha)}\Big(\inf_{n_j\le n\le n_{j+1}} \sqrt{n}\frac{W_n^{(\alpha)}}{D_n^{(\alpha)}} < (1-\varepsilon)\theta\Big) < \infty, \tag{4.7}$$

from which the Borel-Cantelli lemma yields (4.5).

To prove (4.7), let $\widehat{\mathcal{F}}_n := \mathcal{F}_n \vee \mathcal{G}_n$, where $\mathcal{G}_n$, defined in Section 2, denotes the σ-fields generated by the spine up to generation n. Then $\mathbb{Q}^{(\alpha)}$-a.s.,

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}^{(\alpha)}}\Big[\frac{1}{R_\alpha(V(\xi_{n_{j+1}}))}\Big|\,\widehat{F}_n\Big] &= \mathbb{E}_{\mathbb{Q}^{(\alpha)}_{V(\xi_n)}}\Big[\frac{1}{R_\alpha(V(\xi_{n_{j+1}-n}))}\Big] \\
&= \frac{1}{R_\alpha(V(\xi_n))}\mathbb{P}_{V(\xi_n)}\Big(\underline{S}_{n_{j+1}-n} \ge -\alpha\Big) \qquad \text{(by (2.8))} \\
&\le \frac{1}{R_\alpha(V(\xi_n))}.
\end{aligned}$$

It follows that for all $n \le n_{j+1}$,

$$\mathbb{E}_{\mathbb{Q}^{(\alpha)}}\Big[\frac{1}{R_\alpha(V(\xi_{n_{j+1}}))}\Big|\,\mathcal{F}_n\Big] \le \mathbb{E}_{\mathbb{Q}^{(\alpha)}}\Big[\frac{1}{R_\alpha(V(\xi_n))}\Big|\,\mathcal{F}_n\Big] = \frac{W_n^{(\alpha)}}{D_n^{(\alpha)}},$$

where the last equality comes from Lemma 4.2 in [5]. Consequently for all $n_j \le n \le n_{j+1}$,

$$\sqrt{n}\,\frac{W_n^{(\alpha)}}{D_n^{(\alpha)}} \ge Y_n := \sqrt{n_j}\,\mathbb{E}_{\mathbb{Q}^{(\alpha)}}\Big[\frac{1}{R_\alpha(V(\xi_{n_{j+1}}))}\Big|\,\mathcal{F}_n\Big].$$

Remark that $(Y_n, n_j \le n \le n_{j+1})$ is a martingale with mean $\mathbb{E}_{\mathbb{Q}^{(\alpha)}}(Y_{n_j}) = \sqrt{n_j}\,\mathbb{E}_{\mathbb{Q}^{(\alpha)}}(\frac{1}{R_\alpha(S_{n_j})}) \ge (1-\varepsilon)\theta$. The Doob L^2-inequality implies that

$$\begin{aligned}
\mathbb{Q}^{(\alpha)}\Big(\max_{n_j\le n\le n_{j+1}} \big|Y_n - \mathbb{E}_{\mathbb{Q}^{(\alpha)}}(Y_{n_j})\big| \ge \frac{\varepsilon}{2}\theta\Big) &\le \frac{4}{\varepsilon^2\theta^2}\mathrm{Var}_{\mathbb{Q}^{(\alpha)}}(Y_{n_{j+1}}) \\
&\le c_{36}\, n_j^{-\delta_2} = c_{36} j^{-3},
\end{aligned}$$

by (4.6) and the fact that $Y_{n_{j+1}} = \sqrt{\frac{n_j}{n_{j+1}}} \frac{W^{(\alpha)}_{n_{j+1}}}{D^{(\alpha)}_{n_{j+1}}}$. Finally for all large j,

$$\mathbb{Q}^{(\alpha)}\Big(\inf_{n_j \le n \le n_{j+1}} \sqrt{n} \frac{W_n^{(\alpha)}}{D_n^{(\alpha)}} < (1-\varepsilon)\theta \Big) \le \mathbb{Q}^{(\alpha)}\Big(\max_{n_j \le n \le n_{j+1}} \big| Y_n - \mathbb{E}_{\mathbb{Q}^{(\alpha)}}(Y_{n_j}) \big| \ge \frac{\varepsilon}{2}\theta \Big) \le c_{36}\, j^{-3},$$

proving (4.7) and then completing the proof of Proposition 1.3. □

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